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# Random generations of the countable random graph

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## Abstract

We consider random processes more general than those considered by Erdős and Rényi for generating the countable random graph. It is proved that, in the category sense, almost all random processes we consider generate the countable random graph with probability 1. Under a weak boundedness assumption we give a criterion for the random processes which generate the countable random graph almost surely. We also consider further questions asked by Jackson regarding the outcome graphs when the process fails to produce the countable random graph.

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## 1. Introduction

In 1963, Erdős and Rényi [4] discovered the existence of a curious infinite graph  $R$  now known as the countable random graph. Their theorem reads as follows (as in [2]):

**Theorem 1.1.** *If a countable graph is chosen at random, by selecting edges independently with probability  $\frac{1}{2}$  from the set of 2-element subsets of the vertex set, then with probability 1 the resulting graph is isomorphic to  $R$ .*

More formally, we can assume the vertex set to be  $\omega$  and assign the independent probabilities

$$p_{ii} = 0; \quad p_{ij} = p_{ji} = \frac{1}{2}, \quad i \neq j.$$

The space of the resulting graphs can be regarded as the Cantor space  $2^\omega$ , with each element of  $\omega$  coding a 2-element subset of  $\omega$ . Then the random process defined above induces the standard Lebesgue measure on  $2^\omega$ . Thus in the conclusion “with probability 1” refers to this Lebesgue measure on the Cantor space.

In this paper, the main question we would like to investigate is: when does the theorem remain true if the random process is changed?

Of course, if we allow 0 and 1 entries off the diagonal of the infinite probability matrix  $(p_{ij})$  then any countable graph (at least those with infinitely many edges) can be almost surely generated. To avoid this triviality we will assume  $0 < p_{ij} < 1$  for all  $i, j \in \omega$ . Otherwise, the probabilities are still assumed to be independent.

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The consideration of this problem was inspired by Vershik's very interesting results on random and universal metric spaces [6,7]. One can ask questions for random metric spaces similar to the ones we ask for random graphs here. However, we have no good solutions for the general questions. Cameron's papers [2,3] are excellent sources of information on the countable random graph. In fact, we would like to thank both Vershik and Cameron for giving the insightful talks at the Conference for Logic, Algebra, and Geometry in St. Petersburg, which are the main motivations of our investigation.

## 2. Probabilities of first order facts

Let  $L = \{E\}$  be the language of graphs, where  $E$  is a binary relation symbol. For  $n, m \in \omega$  we define a first order sentence  $\varphi_{n,m}$  as follows. First, for any variables  $x_1, \dots, x_n, y_1, \dots, y_m$ , let  $\theta_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)$  be the formula expressing that all of  $x_i$  and  $y_j$  are distinct. Then define the formula  $\psi_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)$  to be

$$\exists z(E(x_1, z) \wedge \dots \wedge E(x_n, z) \wedge \neg E(y_1, z) \wedge \dots \wedge \neg E(y_m, z)).$$

Finally let  $\varphi_{n,m}$  be the universal closure of

$$\theta_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m) \rightarrow \psi_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m).$$

Let  $\Phi = \{\varphi_{n,m} \mid n, m \in \omega\}$ . Then  $\Phi$  is a set of axioms for the theory of the countable random graph. (In [2] it is called the I-property; here we merely want to make it explicit that the I-property is essentially a first order theory.)

With some abuse of the notation, we let

$$\mathcal{E}_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m) = \{G \in 2^\omega \mid G \models \psi_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)\}.$$

Here any  $G \in 2^\omega$  is interpreted as a countable graph with vertex set  $\omega$ . Hence it makes sense to talk about the satisfaction relation when the variables  $x_1, \dots, x_n, y_1, \dots, y_m$  are assigned values in  $\omega$ . Also we let

$$\mathcal{F}_{n,m} = \{G \in 2^\omega \mid G \models \varphi_{n,m}\}.$$

Then

$$\mathcal{F}_{n,m} = \bigcap_{\text{distinct } x_1, \dots, x_n, y_1, \dots, y_m} \mathcal{E}_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m).$$

Finally

$$\{G \in 2^\omega \mid G \cong R\} = \{G \in 2^\omega \mid G \models \Phi\} = \bigcap_{n,m \in \omega} \mathcal{F}_{n,m}.$$

Now for any Borel probability measure  $\mu$  on  $2^\omega$ , we get immediately that

$$\mu(\{G \in 2^\omega \mid G \cong R\}) = 1$$

iff

$$\mu(\mathcal{E}_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)) = 1$$

for all  $n, m \in \omega$  and distinct  $x_1, \dots, x_n, y_1, \dots, y_m \in \omega$ .

If  $\mu$  is the product measure obtained by assigning the independent probabilities  $p_{ij}$ ,  $0 < p_{ij} < 1$ , as we indicated in the setup of our problem, then replacing the existential quantifier  $\exists z$  in the formula  $\psi_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)$  by  $\neg \forall z \neg$ , and noticing that the matrix of the formula is a conjunction of independent events, we get that

$$\mu(\mathcal{E}_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)) = 1 - \prod_{z=0}^{\infty} (1 - p_{x_1 z} \dots p_{x_n z} (1 - p_{y_1 z}) \dots (1 - p_{y_m z})). \quad (1)$$

This formula will be the basis of further computations leading to our theorems in the subsequent sections.

For the rest of this section we draw a few corollaries and make some remarks.

First of all, the above is essentially an abstraction of the proof of [Theorem 1.1](#). In the case of [Theorem 1.1](#)  $\mu = \lambda$  is the Lebesgue measure on  $2^\omega$ , and

$$\lambda(\mathcal{E}_{n,m}(x_1, \dots, x_m, y_1, \dots, y_n)) = 1 - \prod_{z=0}^{\infty} \left(1 - \left(\frac{1}{2}\right)^{n+m}\right) = 1.$$

The same proof goes through with more relaxed conditions on the probabilities  $(p_{ij})$ . For instance, it is well known that when the probabilities  $\frac{1}{2}$  are replaced by an arbitrarily fixed  $0 < p < 1$ , the resulting graph from the random process is isomorphic to  $R$  with probability 1. It seems that the most general setting for this to happen is described in the following corollary.

**Theorem 2.1.** *Let  $(p_{ij})$  be a probability matrix (i.e. satisfying  $p_{ii} = 0$  and  $0 < p_{ij} = p_{ji} < 1$ ,  $i \neq j$ ). Suppose that there are  $0 < A < B < 1$  such that  $A < p_{ij} < B$  for all  $i \neq j$ . Let  $\mu$  be the product measure on  $2^\omega$  generated by  $(p_{ij})$ . Then  $\mu(\{G \in 2^\omega \mid G \cong R\}) = 1$ .*

This is because, by [Eq. \(1\)](#) and our assumption, we have that

$$\mu(\mathcal{E}_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)) \geq 1 - \prod_{z=0}^{\infty} (1 - A^n(1 - B)^m) = 1.$$

Thus even when the probabilities  $p_{ij}$  are fairly arbitrary, as long as they are bounded away from 0 and from 1, the countable random graph will be the outcome of the random process with probability 1.

Another remark we would like to make is that the computation leading to [Eq. \(1\)](#) cannot be easily generalized for arbitrary first order formulas or sentences. For instance, consider the formula

$$\rho(0, 1, 2) = \forall z((E(z, 0) \wedge E(z, 1) \wedge \neg E(0, 1) \wedge \neg E(0, 2) \wedge \neg E(1, 2)) \vee E(z, 2)).$$

There is no simple expression for the measure

$$\mu(\{G \in 2^\omega \mid G \models \rho(0, 1, 2)\}),$$

since the events in the matrix are not independent. However, [Theorem 2.1](#) abstractly implies that, under the boundedness assumption for the probabilities, for any sentence  $\varphi$  even in  $L_{\omega_1\omega}$ , the measure  $\mu(\{G \in 2^\omega \mid G \models \varphi\})$  is either 0 or 1. In fact, the measure is 1 when  $\varphi$  is true in  $R$  and is 0 otherwise.

Recall that the infinite permutation group  $S_\infty$  acts naturally on the space of  $L$ -models with universe  $\omega$  [1]. The orbit equivalence relation is exactly the isomorphism relation of such  $L$ -models. In our setup here, an action of  $S_\infty$  on  $2^\omega$  is induced (it is not merely the permutation of coordinates). In the case that all  $p_{ij}$  are a constant ( $\frac{1}{2}$  or some fixed  $p$ ) the induced measure  $\mu$  on  $2^\omega$  is invariant under this action of  $S_\infty$ . Invariance does not hold under the boundedness assumption of [Theorem 2.1](#), but from our remarks above we get that the measure  $\mu$  is ergodic under the action of  $S_\infty$ . This is because, by a theorem of Lopez-Escobar [5], invariant Borel sets under this action of  $S_\infty$  are given exactly by the  $L_{\omega_1\omega}$  sentences.

**Corollary 2.2.** *Let  $(p_{ij})$  be a probability matrix and suppose there are  $0 < A < B < 1$  with  $A < p_{ij} < B$  for all  $i \neq j$ . Let  $\mu$  be the probability measure induced by  $(p_{ij})$ . Then  $\mu$  is ergodic with respect to the action of  $S_\infty$  on  $2^\omega$  inducing the isomorphism relation of countable graphs. (Compare [6,7].)*

### 3. A criterion for randomness

In light of [Theorem 2.1](#) we will continue our inquiry with focus on the nontrivial case when the probabilities  $p_{ij}$  are not bounded away from 0 or from 1. The general case is still too complicated, so we make some further simplification assumptions. We will assume that for all but finitely many  $i$ , the probabilities  $p_{ij}$  are bounded away from 0 and from 1. Without loss of generality, we assume that there are  $0 < A < B < 1$  and  $N \in \omega$  such that  $A < p_{ij} < B$  for all  $i, j > N$  and  $j \neq i$ . The problem is to describe a condition on  $(p_{ij})$  so as to characterize the case  $\mu(\{G \in 2^\omega \mid G \cong R\}) = 1$ .

In the first pass of the investigation we will work with the following additional assumptions:  $N = 0$  and  $p_{0j} \rightarrow 0$  as  $j \rightarrow \infty$ . For notational clarity we will denote  $p_{0j}$  by  $\alpha_j$ .

By the computation of the preceding section we get that

$$\mu(\{G \in 2^\omega \mid G \cong R\}) = 1$$

iff for all  $m, n \in \omega$  and distinct  $x_1, \dots, x_n, y_1, \dots, y_m \in \omega$ ,

$$\mu(\mathcal{E}_{n,m}(x_1, \dots, x_n, y_1, \dots, y_m)) = 1,$$

or

$$\prod_{z=0}^{\infty} (1 - p_{x_1 z} \dots p_{x_n z} (1 - p_{y_1 z}) \dots (1 - p_{y_m z})) = 0.$$

If  $x_1, \dots, x_n, y_1, \dots, y_m > 0$  then our boundedness assumption and the proof of [Theorem 2.1](#) give the desired equality. It follows that

$$\mu(\{G \in 2^\omega \mid G \cong R\}) = 1$$

iff for all  $m, n \in \omega$  and distinct  $x_1, \dots, x_n, y_1, \dots, y_m > 0$ , both

$$\mu(\mathcal{E}_{n+1,m}(0, x_1, \dots, x_n, y_1, \dots, y_m)) = 1 \quad (2)$$

and

$$\mu(\mathcal{E}_{n,m+1}(x_1, \dots, x_n, 0, y_1, \dots, y_m)) = 1. \quad (3)$$

Since  $\alpha_z \rightarrow 0$  as  $z \rightarrow \infty$ , there is some  $C < 1$  such that  $\alpha_z \leq C$  for all  $z$ . And Eq. (3) is equivalent to

$$\prod_{z=0}^{\infty} (1 - p_{x_1 z} \dots p_{x_n z} (1 - \alpha_z) (1 - p_{y_1 z}) \dots (1 - p_{y_m z})) = 0,$$

which is therefore true since

$$\prod_{z=0}^{\infty} (1 - p_{x_1 z} \dots p_{x_n z} (1 - \alpha_z) (1 - p_{y_1 z}) \dots (1 - p_{y_m z})) \leq \prod_{z=0}^{\infty} (1 - A^n (1 - B)^m (1 - C)) = 0.$$

Eq. (2), on the other hand, is equivalent to

$$\prod_{z=0}^{\infty} (1 - \alpha_z p_{x_1 z} \dots p_{x_n z} (1 - p_{y_1 z}) \dots (1 - p_{y_m z})) = 0. \quad (4)$$

From the boundedness assumption, an upper bound of the expression is

$$\prod_{z=0}^{\infty} (1 - A^n (1 - B)^m \alpha_z),$$

and a lower bound is

$$\prod_{z=0}^{\infty} (1 - B^n (1 - A)^m \alpha_z).$$

Both bounds are of the format

$$\prod_{z=0}^{\infty} (1 - a \alpha_z)$$

for some  $0 < a < 1$ .

To further analyze the expression we take logarithm of the expression and use the McLaurin series

$$\log(1 - ax) = - \sum_{k=1}^{\infty} \frac{1}{k} a^k x^k$$

to get

$$\log \prod_{z=0}^{\infty} (1 - a\alpha_z) = - \sum_{z=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} a^k \alpha_z^k,$$

and note that

$$\prod_{z=0}^{\infty} (1 - a\alpha_z) = 0 \quad (5)$$

iff the series

$$\sum_{z=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} a^k \alpha_z^k$$

diverges.

Since all terms of the last series are positive, the convergence of the series is equivalent to the convergence of

$$\sum_{k=1}^{\infty} \frac{1}{k} a^k \left( \sum_{z=0}^{\infty} \alpha_z^k \right). \quad (6)$$

Let  $s_k = \sum_{z=0}^{\infty} \alpha_z^k$ . Then we note that  $s_k \leq s_1$  for all  $k \geq 1$  since  $0 < \alpha_z < 1$  for all  $z$ . This means that if  $s_1$  is convergent then so are all  $s_k$  and also

$$\sum_{k=1}^{\infty} \frac{1}{k} a^k \left( \sum_{z=0}^{\infty} \alpha_z^k \right) \leq s_1 \sum_{k=1}^{\infty} \frac{1}{k} a^k,$$

which is convergent since  $|a| < 1$ . Conversely, since

$$\sum_{k=1}^{\infty} \frac{1}{k} a^k \left( \sum_{z=0}^{\infty} \alpha_z^k \right) \geq a s_1,$$

if  $s_1$  is divergent so is the series under consideration. We have thus established that the following statements are equivalent:

- (i) Eq. (5) holds for any  $0 < a < 1$ ;
- (ii) Eq. (5) holds for some  $0 < a < 1$ ;
- (iii) series (6) is divergent for any  $0 < a < 1$ ;
- (iv) series (6) is divergent for some  $0 < a < 1$ ;
- (v)  $s_1$  is divergent.

From the equivalence of (i) and (ii) we get that they are also equivalent to Eq. (4). We summarize this in the following theorem.

**Theorem 3.1.** *Let  $(p_{ij})$  be a probability matrix satisfying that*

- (a) *there are  $0 < A < B < 1$  such that  $A < p_{ij} < B$  for all  $i, j > 0$  and  $j \neq i$ , and*
- (b)  *$\alpha_j = p_{0j} \rightarrow 0$  as  $j \rightarrow \infty$ .*

*Let  $\mu$  be the probability measure induced by  $(p_{ij})$ . Then  $\mu(\{G \in 2^\omega \mid G \cong R\}) = 1$  iff  $\sum_{j=0}^{\infty} \alpha_j = \infty$ .*

Condition (b) was only used in the argument for Eq. (3). If condition (b) is eliminated, then the discussion for Eq. (2) can be repeated verbatim and a discussion for Eq. (3) is similar (with  $\alpha_z$  replaced by  $1 - \alpha_z$ ). Hence it is straightforward to obtain the following slightly general theorem.

**Theorem 3.2.** *Let  $(p_{ij})$  be a probability matrix and  $\mu$  the induced probability measure on  $2^\omega$ . Suppose there are  $0 < A < B < 1$  such that  $A < p_{ij} < B$  for all  $i, j > 0$  and  $j \neq i$ . Then  $\mu(\{G \in 2^\omega \mid G \cong R\}) = 1$  iff both  $\sum_{j=0}^{\infty} \alpha_j = \infty$  and  $\sum_{j=0}^{\infty} (1 - \alpha_j) = \infty$ .*

This criterion was somewhat surprising when we first obtained it, since it is a bit anti-intuitive in some concrete situations. For instance, the criterion guarantees that the countable random graph will be generated with probability 1 if, other than the weak boundedness condition, there are subsequences  $p_{0j_k} \rightarrow 0$  and  $p_{0j_l} \rightarrow 1$  simultaneously, since in this case both summability considered would fail.

One can run the same proof a third time to address the general case  $N > 0$ . We give the concluding criterion without further proof.

**Theorem 3.3.** *Let  $(p_{ij})$  be a probability matrix and  $\mu$  the induced probability measure on  $2^\omega$ . Suppose there are  $0 < A < B < 1$  and  $N \in \omega$  such that  $A < p_{ij} < B$  for all  $i, j > N$  and  $j \neq i$ . Then the following are equivalent:*

- (I)  $\mu(\{G \in 2^\omega \mid G \cong R\}) = 1$ ;
- (II) for any  $\epsilon_0, \dots, \epsilon_N \in \{+1, -1\}$ ,

$$\sum_{j=0}^{\infty} q_{0j}^{(\epsilon_0)} \dots q_{Nj}^{(\epsilon_N)} = \infty,$$

where

$$q_{ij}^{(\epsilon)} = \frac{1 + \epsilon}{2} + \epsilon p_{ij} = \begin{cases} p_{ij}, & \text{if } \epsilon = 1, \\ 1 - p_{ij}, & \text{if } \epsilon = -1. \end{cases}$$

One can suitably code the product measures in a Polish space. For instance, each measure can be coded by an infinite sequence of numbers, thus an element of  $(0, 1)^\omega$ . The criterion we obtained is transparently a dense  $G_\delta$  condition, since summability is  $F_\sigma$ . In fact, this observation can be fully generalized to condition (1). Since the transformations (taking logarithm, subtractions and multiplications) are all continuous, we see that condition (1) is dense  $G_\delta$ . Thus we have obtained the following theorem.

**Theorem 3.4.** *The set of probability matrices such that for the induced measure  $\mu$ ,  $\mu(\{G \in 2^\omega \mid G \cong R\}) = 1$ , is dense  $G_\delta$ .*

Therefore the Erdős–Rényi theorem is true for (in the category sense) almost all random processes with independent probabilities assigned for the edges.

#### 4. Further inquiries

In the preceding section we worked with a weak boundedness assumption and obtained a criterion for a random process to generate the countable random graph with probability 1. Jackson asked if the criterion is not met, namely the countable random graph is not almost surely the outcome, what kind of outcomes we should expect. Of course there could be at most countably many possible outcomes each with a positive probability. But could there be uncountably many outcomes with a positive probability collectively but zero probability individually? How many first order theories can these outcomes possess? Could there be only one, or finitely many, or countably many, possible theories possessed by a collection of outcomes of full measure? These questions are intriguing and we will attempt to answer them in this section, at least for the situation of our setup.

Again, we treat the simplest case first. Assume that there are  $0 < A < B < 1$  such that  $A < p_{ij} < B$  for  $i, j > 0$  and  $j \neq i$ . In light of Theorem 3.1, assume  $\sum_{j=0}^{\infty} \alpha_j < \infty$ , where  $\alpha_j = p_{0j}$ . By Theorem 2.1, the resulting subgraph with vertex set  $\omega \setminus \{0\}$  is isomorphic to the countable random graph.

It turns out that Jackson's questions have simple answers in this setup; we will claim that almost surely the vertex 0 has finite degree. To begin with, consider the formula

$$\gamma_0(0) = \forall z (\neg z = 0 \rightarrow \neg E(z, 0)).$$

It is easy to see that

$$\mu_0 = \mu(\{G \in 2^\omega \mid G \models \gamma_0(0)\}) = \prod_{z=1}^{\infty} (1 - \alpha_z).$$

By taking logarithm and using the McLaurin series of  $\log(1 - x)$  again we get that

$$\log \mu_0 = - \sum_{z=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} \alpha_z^k.$$

In the notation we used in the preceding section,

$$\log \mu_0 = - \sum_{k=1}^{\infty} \frac{1}{k} s_k.$$

Now by our assumption,  $s_1$  is convergent, and consequently so are all  $s_k$  for  $k \geq 1$ . It is necessary that  $\alpha_z \rightarrow 0$  as  $z \rightarrow \infty$ . Therefore there is  $0 < C < 1$  such that  $\alpha_z < C$  for all  $z$ . Notice that

$$s_{k+1} = \sum_{z=1}^{\infty} \alpha_z^{k+1} \leq C \sum_{z=1}^{\infty} \alpha_z^k = C s_k.$$

Thus by the ratio test the series for  $\log \mu_0$  is convergent. This implies that  $\mu_0 > 0$ .

Now fix a general  $N > 0$  and consider the property that  $\neg E(z, 0)$  for all  $z > N$ . This is a first order property in the parameters 0 through  $N$ , which we denote (and abbreviate) by  $\gamma_N(0)$ . It is straightforward to get

$$\mu_N = \mu(\{G \in 2^\omega \mid G \models \gamma_N(0)\}) = \prod_{z=N+1}^{\infty} (1 - \alpha_z).$$

Thus

$$\mu_N = \left( \prod_{z=1}^N (1 - \alpha_z) \right)^{-1} \mu_0 > 0.$$

Letting  $N \rightarrow \infty$ , we get that  $\mu_N \rightarrow 1$ . Let  $T(0)$  be the first order type axiomatizing that 0 has infinite degree. Our computation implies that  $\mu(\{G \in 2^\omega \mid G \models T(0)\}) = 0$ . Thus almost surely 0 has finite degree in the outcome of the random process. We will denote  $\deg(0)$  for the degree of 0 in an outcome.

It is clear from our computations above that the event  $\deg(0) = n$  has a positive probability for all  $n \in \omega$ . Outcomes in which 0 has different degrees are obviously non-isomorphic. For a fixed  $n \in \omega$ , there are only countably many possible outcomes with  $\deg(0) = n$ . Moreover the type of 0 determines the isomorphism type of the outcome graph. This is because the countable random graph is (ultra)homogeneous, that is, if two finite subgraphs  $F_1$  and  $F_2$  of  $R$  are partially isomorphic then the partial isomorphism can be extended to an isomorphism of  $R$  onto  $R$ . Suppose we have two outcomes  $G_1$  and  $G_2$ , both with  $\deg(0) = n$ , and denoting  $F_1$  the subgraph of  $G_1$  consisting of all vertices adjacent to 0 and  $F_2$  the respective subgraph of  $G_2$ , suppose also that  $F_1$  is partially isomorphic to  $F_2$ . Then the extended isomorphism of  $G_1 \setminus \{0\}$  onto  $G_2 \setminus \{0\}$  induces an isomorphism of  $G_1$  onto  $G_2$ . This argument shows that there are only countably many non-isomorphic one-point extensions of the countable random graph if the extra vertex has finite degree, and they correspond faithfully to their theories.

Let us summarize the answers to Jackson's questions. Let  $R_0, R_1, \dots, R_n, \dots$  enumerate all non-isomorphic one-point extensions of  $R$  with  $\deg(0)$  finite. For each  $n \in \omega$ , the probability of the outcome being isomorphic to  $R_n$  is positive. There are no uncountable collections of outcomes with positive probabilities collectively and zero probabilities individually. Each  $R_n$  has a distinctive first order theory. Thus the answers for first order theories generated are the same as for the isomorphism types. In particular, it is impossible to have finitely many theories whose models constitute a collection of outcomes of full measure.

With the more relaxed weak boundedness assumption, namely the assumption of [Theorem 3.3](#), we have similar answers to the questions. Again there are countably many non-isomorphic, finite, finite-degree extensions of the countable random graph. With probability 1 the outcome is one of them. Each of them is generated with a positive probability. And they have distinctive first order theories.

In either case, it is (trivial but) interesting to note that the countable random graph is generated with probability 0.

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